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Behavior of the Chebyshev Operator of Best Approximation from a Curve of Functions

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The local behavior of the Chebyshev operator of best approximation from a curve of functions (of which exponential sums are a special case) is studied, with emphasis on local existence.

Let ψ be continuous on the open interval (μ, ν) (which may be infinite). Let $[\alpha, \beta]$ be a closed finite interval. Let (γ, δ) be an open interval such that $ax \in (\mu, \nu)$ for $a \in (\gamma, \delta)$ and $x \in [\alpha, \beta]$. Let $n > 0$, $m \geqslant 0$ and let $V_{n,m}(\psi)$ be the set of functions of the form

$$F(A, x) = \sum_{k=1}^{n} a_k \psi(a_{n+k} x) + \sum_{k=1}^{m} a_{2n+k} x^{k-1} \qquad a_{n+k} \in (\gamma, \delta).$$

A classical set of functions of this form is the set of exponential sums $V_{n,0}(\exp)$, with $(\mu, \nu) = (\gamma, \delta) = (-\infty, \infty)$. General families of the form $V_{n,0}(\psi)$ were first studied by Hobby and Rice [4].

There is no loss of generality in requiring that $a_{n+1}, \ldots, a_{2n}$ be distinct and that all be nonzero if $m > 0$, which we assume henceforth. Define the *degeneracy* of F at A, denoted by $d(A)$, to be the number of zeros in $\{a_1, \ldots, a_n\}$. In many cases of interest, F is varisolvent (see Rice [5, 3 ff] for the definition and the basic theory) and, in particular, F is unisolvent of degree $2n + m - d(A)$ at A. We henceforth assume that this is the case. Consider the Chebyshev approximation of $f \in C[\alpha, \beta]$ by $V_{n,m}(\psi)$. A best approximation is characterized by alternation of its error curve and is unique. Denote the best approximation to f (if it exists) by $T(f)$. Only in very simple cases do best approximations exist to all $f \in C[\alpha, \beta]$. In particular, with $\psi = \exp$, global existence occurs only for $n = 1$ and $m = 0$, that is, only in the case where $F(A, x) = a_1 \exp(a_2 x)$. However, there is a local existence result due to the author [1].

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THEOREM 1. *Let the best approximation $F(A, \cdot)$ to f be of maximum degree (that is, $d(A) = 0$). There is a neighborhood of f in $C[\alpha, \beta]$ such that any element of that neighborhood has a best approximation. $\{f_k\} \to f$ implies $\{T(f_k)\} \to T(f)$.*

In the above theorem (and subsequently) $\to$ denotes uniform convergence on $[\alpha, \beta]$. No serious study has been made of what happens to existence when best approximations are not of maximum degree, except for the analysis of Schmidt [6, p. 170] of the case in which f is a degenerate approximation.

DEFINITION. $V_{n,m}(\psi)$ is *m-empty* if no sum of an element of $V_{n,0}(\psi)$ and a polynomial of exact degree $\geqslant m$ is in $V_{n,m}(\psi)$.

THEOREM 2. *Let $V_{n,m}(\psi)$ be m-empty. Let $V_{1,m}(\psi)$ contain a sequence $\{g_j\} \to p$, a polynomial having exact degree $\geqslant m$. Let f be an approximation of less than maximum degree. There exists $\{f_k\}$ converging uniformly to f with f_k having no best approximation from $V_{n,m}(\psi)$.*

Proof. Let $f = F(A, \cdot)$ and $f_k = F(A, \cdot) + p/k$. We have

$$h_{jk} = F(A, \cdot) + g_j/k \in V_{n,m}(\psi)$$

and

$$\| f_k - h_{jk} \| \to 0.$$

But $f_k \notin V_{n,m}(\psi)$.

THEOREM 3. *Let $V_{n,m+1}(\psi)$ also be varisolvent with an element having degeneracy ℓ being of degree $2n + m + 1 - \ell$. Let $V_{n,m}(\psi)$ be m-empty. Let $V_{1,m}(\psi)$ contain a sequence $\{g_j\} \to p$, a polynomial of exact degree m. Let $d(A) > 0$. Let $f - F(A, \cdot) \not\equiv 0$ alternate at least $2n + m + 1 - d(A)$ times. There exists $\{f_k\} \to f$ with f_k having no best approximation from $V_{n,m}(\psi)$.*

Proof. Define

$$f_k = f + p/k \qquad h_k = F(A, \cdot) + p/k;$$

then $f - F(A, \cdot) = f_k - h_k$ alternates $2n + m + 1 - d(A)$ times. $\{f_k - F(A, \cdot) - g_j/k\} \to f_k - h_k$; hence if a best approximation $F(B, \cdot)$ exists to f_k from $V_{n,m}(\psi)$

$$\| f_k - F(B, \cdot) \| \leqslant \| f_k - h_k \|.$$

By an argument due to de la Vallée-Poussin [2, p. 226], this implies that $F(B, \cdot) - h_k$ has $2n + m + 1 - d(A)$ zeros, counting double zeros twice. But $h_k \in V_{n,m+1}(\psi)$ and has degree $2n + m + 1 - d(A)$. Therefore, a difference of it and $F(B, \cdot) \in V_{n,m+1}(\psi)$ can have at most $2n + m - d(A)$ zeros, counting double zeros twice, or $F(B, \cdot) \equiv h_k$ [5, 4]. But $h_k \notin V_{n,m}(\psi)$ and we have a contradiction.

What happens when exactly $2n + m - d(A)$ alternations occur is not known in general. However, from [3, p. 107] we have

THEOREM 4. *Let $d(A) > 0$ and $f - F(A, \cdot)$ alternate exactly $2n + m - d(A)$ times. There exists $\{f_k\} \to f$ such that no subsequence $\{f_{k(j)}\}$ has existence of best approximations and uniform convergence of best approximations from $V_{n,m}(\psi)$ to $F(A, \cdot)$.*

Following Schmidt [6], we could define the Chebyshev operator T to be continuous at f if (i) best approximations exist in a neighborhood of f and (ii) $\{f_k\} \to f$ implies $T(f_k) \to T(f)$. Combining Theorems 1–4, we get

THEOREM 5. *Let $V_{n,m}(\psi)$ be m-empty. Let $V_{1,m}(\psi)$ have a polynomial p of exact degree m as a limit point. Let $V_{n,m+1}(\psi)$ be varisolvent with elements of degeneracy ℓ being of degree $2n + m + 1 - \ell$. T is continuous at f if and only if $T(f)$ exists and is of maximum degree.*

In Theorems 2, 3, 5, a hypothesis was that $V_{1,m}(\psi)$ had a limit point p, a polynomial of degree m or more. A sufficient condition for this to occur is given by the following theorem.

THEOREM 6. *Let ψ have a Taylor series $\sum_{k=0}^{\infty} a_k x^k$ convergent in a neighborhood of zero and l be the lowest index $k \geqslant m$ such that $a_k \neq 0$. Any polynomial of the form $ax^\ell + p(x)$, p of degree $m - 1$, is a limit point of $V_{1,m}(\psi)$.*

Proof. Let $p_j(x)$ be the polynomial obtained by truncating the Taylor series for $(aj^\ell/a_\ell)\, \psi(x/j)$ at degree $m - 1$, then

$$(aj^\ell/a_\ell)\, \psi(x/j) - p_j(x) + p(x) = ax^\ell + p(x) + ar_j(x),$$

where

$$r_j(x) = \left(\sum_{k=\ell+1}^{\infty} a_k x^k j^{\ell-k} \right) \Big/ a_\ell .$$

We have

$$\begin{aligned} | a_\ell r_j(x)| &\leqslant \sum_{k=\ell+1}^{\infty} | a_k x^k j^{\ell-k} | \\ &\leqslant | a_{\ell+1} x^{\ell+1} |/j + \sum_{k=\ell+2}^{\infty} | a_k (x/j^{1/(\ell+2)})^k |/j. \end{aligned}$$

For $x \in [\alpha, \beta]$ and all j sufficiently large $x/j^{1/(\ell-2)}$ is in the region of convergence of ψ, hence the right-hand side tends to zero uniformly on $[\alpha, \beta]$ as $j \to \infty$.

Theorems 2 and 6 show that the existence of best approximations to all $f \in C[\alpha, \beta]$ can occur only in very simple cases. For example with $\psi(x) = \exp(x)$, existence is guaranteed only when $n = 1$, $m = 0$.

We remarked earlier that what happens to nearby existence when $d(A) > 0$ and $f - F(A, \cdot)$ alternates exactly $2n + m - d(A)$ times is unknown in general. The following example shows that nonexistence nearby need not occur even if nonexistence occurs globally.

EXAMPLE. Let $[\alpha, \beta] = [0, 1]$ and let $f(x) = T_2^*(x) = 8x^2 - 8x + 1$, the second Chebyshev polynomial on [0, 1]. Approximate f by $V_{1,1}(\psi)$, where $\psi(x) = \log(1 + x)$, discussed in [7]. As f alternates twice on [0, 1], 0 is the unique best approximation to f. Suppose g exists near f with no best approximation from $V_{1,1}(\psi)$. As best approximations by $H = V_{1,1}(\psi) \cup \{cx + d\}$ exist to all elements of $C[0, 1]$ by [8], g has a best approximation in H, which must, therefore, be a first degree polynomial $cx + d$. By the characterization of best approximations by H in [8], $g(x) - cx - d$ alternates at least three times (the amplitude must be close to 1 by standard results on continuity of the error functional). But $f(x) - cx - d$ is a polynomial of degree two and cannot approximately alternate three times with amplitude near 1. We have a contradiction and g does not exist.

Exactly the same situation occurs when we approximate by $V_{1,1}(\psi)$, $\psi = \exp$.

The applicability of the theory of this paper obviously depends on what families $V_{n,m}(\psi)$ are alternating with the required degree. The author has proven that $V_{n,m}(\psi)$ is alternating with the required degree when

(i)	$\psi(x) = 1/(1 - x)$	$m \geq 0$	[12]
(ii)	$\psi(x) = \exp(x)$	$m \geq 0$	[13]
(iii)	$\psi(x) = \log(1 + x)$	$m = 1$	[9]
		$m > 1$	[13]
(iv)	$n = 1$, $m = 0$, ψ varied		[10]
	$n = 1$, $m = 1$, ψ varied		[11]
	$n = 1$, m general, ψ varied		[13]

Varisolvence follows from results of Barrar and Loeb.

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